Strain Compatibility Primer—Part 1

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This article, the first of a two-part series on strain compatibility analysis, introduces strain compatibility concepts, applies them to concrete components, and demonstrates the basics of strain compatibility analysis. Part 2 will be presented in a future issue of *ASPIRE*® and will evaluate methods for estimating the flexural strength of concrete components specified in the American Association of State Highway and Transportation Officials' *AASHTO LRFD Bridge Design Specifications*.[1] That article will discuss the assumptions and limitations of the rectangular stress distribution method as detailed in the AASHTO LRFD specifications, as well as how those limitations can be overcome by using strain compatibility analysis.

Strain and displacement compatibility are fundamental engineering concepts. Many scientific and mathematical symbols and equations are needed to explain them thoroughly, but at the heart of the matter is a simple concept: A solid object subjected to load deforms and the displacements of adjacent points are related to one another. The displacements, and the strains, are said to be compatible. **Figure 1** illustrates compatible and incompatible strains and displacements.

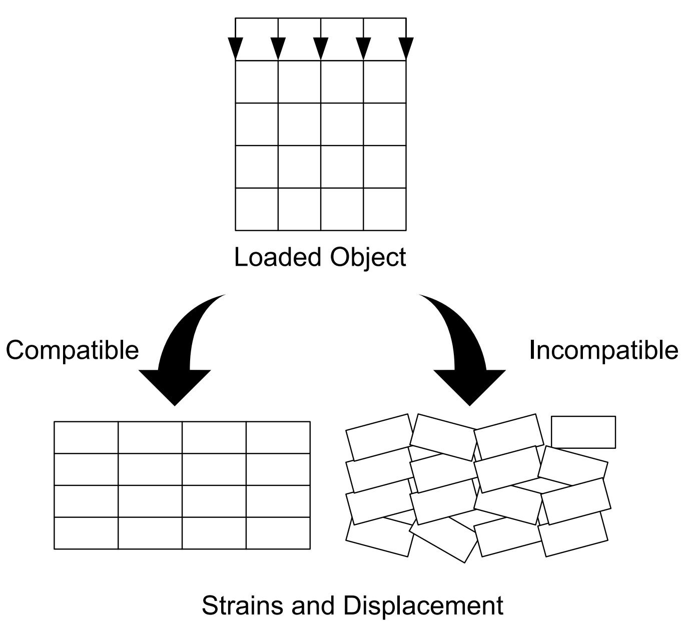

Figure 1. Compatible and incompatible strains and displacements for a loaded object. All Figures and Tables: Richard Brice.

Strain Compatibility Basics

Strain compatibility is fundamental to determining the flexural strength of concrete beams. Bridge engineers are typically familiar with the rectangular stress distribution method for estimating flexural strength specified in the AASHTO LRFD specifications. The equations in the specification are derived from conditions of equilibrium and strain compatibility. Compatibility of strains is the familiar "plane sections remain plane" assumption for linear strain distribution. Reinforcement strain is compatible with concrete strain based on the linear strain distribution assumption and the assumption that materials are bonded together. Strain compatibility ensures that reinforced concrete structures behave predictably under load, which enables mathematical expressions to describe the relationships among displacement, strain, stress, and force.

The rectangular stress distribution method is an excellent predictor of flexural strength for steel-reinforced and prestressed concrete structural elements, so long as the section analyzed conforms to the applicable assumptions and limitations. Key assumptions are that all reinforcement can be considered at a single location, the composite deck and beam are made of the same concrete, and the section can be represented as an idealized T-shape. These assumptions limit the application and accuracy of the rectangular stress distribution method.

In its more general form, strain compatibility can be used to determine the stress and strain for many situations, including concrete crushing strain, steel yield or fracture strain, balanced strain conditions, or any other limiting condition. Geometrically complex shapes with different types of reinforcement throughout as well as different materials with simple linear-elastic to complex nonlinear stress-strain relationships can be analyzed using strain compatibility. Concrete can be unconfined, confined, fiber reinforced, or another type. Reinforcement is typically carbon steel or high-strength prestressing strand, but it could also be high-strength steel, carbon-fiber bars, stainless steel strand, glass-fiber-reinforced-polymer strand, or any other suitable material. Stress-strain relationships must be available for all materials used. Strain compatibility is the "Swiss Army knife" of flexural strength analysis.

Strain compatibility analysis is typically an iterative procedure where the neutral axis location is varied until conditions of equilibrium are satisfied. Calculations are quickly and easily performed with a spreadsheet. The beam section is subdivided into concrete and reinforcing elements. The following calculations are performed at each iteration:

1. Estimate the neutral axis location.
2. Compute the strain at the centroid of each cross-section element assuming a linear strain distribution.
3. Compute the stress for each cross-section element using its stress-strain relationship.
4. Compute the force contribution of each element by multiplying the element's stress by its cross-section area.
5. Sum the force contributions of all cross-section elements.
6. Repeat steps 1 through 5 until the summation of the forces is zero. This is the equilibrium condition.

Once the equilibrium condition has been found, sum the moments of all cross-section element forces about any convenient point (the top of the beam section or the neutral axis is a good candidate). The resulting moment is the strength of the section, provided all stresses and strains have not exceeded the physical limits of the materials.

Some considerations when using strain compatibility:

- Stress-strain relationships for concrete, prestressing steel, nonprestressed steel, and other materials can be found in the *PCI Bridge Design Manual*,[2] textbooks on concrete design, or elsewhere in literature.
- Areas of concrete with different concrete strengths must be placed in separate elements or layers. At the strength limit state, most of the strain over the beam section is beyond the linear-elastic properties of the material. For this reason, do not transform concrete areas based on their moduli of elasticity.
- A concrete area such as a flange or web with chamfered or radiused edges can be modeled as a rectangle having a cross-section area approximately equal to the actual section area.
- Any concrete that is in tension may be cracked, and its force contribution is therefore neglected. However, fiber-reinforced concretes, such as ultra-high-performance concrete, have considerable tensile strength that should be included in the analysis.
- If the rows of steel reinforcement consist of the same material and are somewhat close together, the total steel area can be lumped at its centroid. The stress is found from the stress-strain relationship and the strain at the centroid of the steel. The force is the stress multiplied by the total area. However, if there are different types of steel or layers of steel are not closely spaced, or if more accuracy is desired, the strain, stress, and force in each individual layer or steel type should be computed.
- For low-rupture strain reinforcement, such as stainless steel and carbon-fiber strands, rupture is more likely to occur before concrete reaches its crushing strain. The steps for strain compatibility analysis are the same. The only difference is that the neutral axis location is determined by fixing the strain at the level of reinforcement expected to rupture, rather than the strain of 0.003 at the extreme concrete compression fiber.
- In most cases, the neutral axis for the strength limit state is not in the same location as the neutral axis for the service limit state.

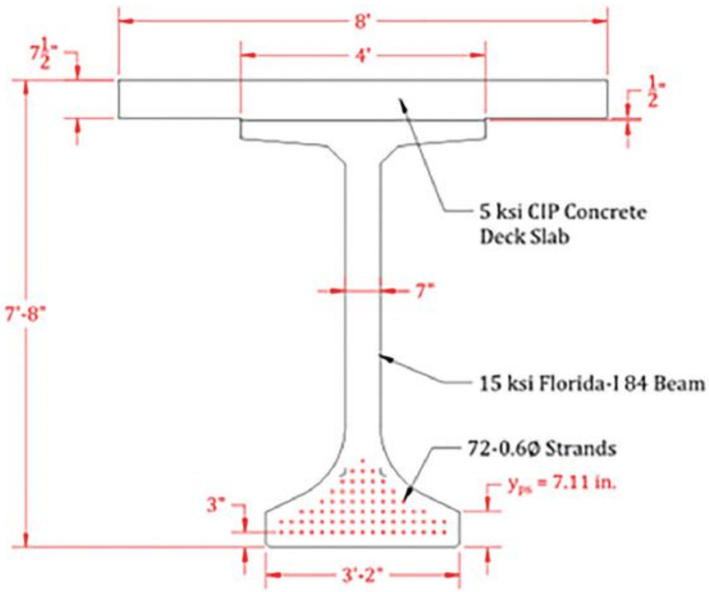

Figure 2. An 84-in.-deep Florida I-beam girder made composite with a 7.5-in.-thick, cast-in-place (CIP) structural deck slab.

- Strain compatibility analysis is typically not appropriate for sections with unbonded reinforcement because strains in the unbonded reinforcement are not compatible with the surrounding concrete.

Example

Consider the prestressed concrete beam and composite cast-in-place deck slab with dimensions, concrete strengths, and reinforcement shown in **Fig. 2**.

A bilinear compression stress-strain relationship is selected for the deck and girder concrete.

$$f_c(\varepsilon_c) = \begin{cases} E_c \varepsilon_c & \text{for } \varepsilon_c < \dfrac{0.85f'_c}{E_c} \\ 0.85f'_c & \text{for } \varepsilon_c \geq \dfrac{0.85f'_c}{E_c} \end{cases}$$

A nonlinear tension stress-strain relationship is selected for the prestressing reinforcement. For this example, the "power formula" is used with properties (f_{pu} = 270 ksi, f_{py} = 243 ksi, and E_s = 28,500 ksi) consistent with ASTM A416 Grade 270 low-relaxation strand.[3]

$$f = 28{,}500\,\varepsilon_s \left[0.031 + \frac{0.969}{\left[1 + (112.77\varepsilon_s)^{7.36}\right]^{\frac{1}{7.36}}} \right] \leq 270 \text{ ksi}$$

The beam cross section is modeled as a set of rectangular concrete layers (**Fig. 3**). The modulus of elasticity of concrete E_c, which is calculated using Eq. (C5.4.2.4-2) of the AASHTO LRFD specifications, is 4287 ksi for the deck and 7425 ksi for the girder.

The calculations for one iteration are as follows:

Assume the neutral axis depth c = 12.0 in.

Assuming that strands are jacked to 75% of f_{pu} and approximately 21% prestress losses occur, the effective prestress

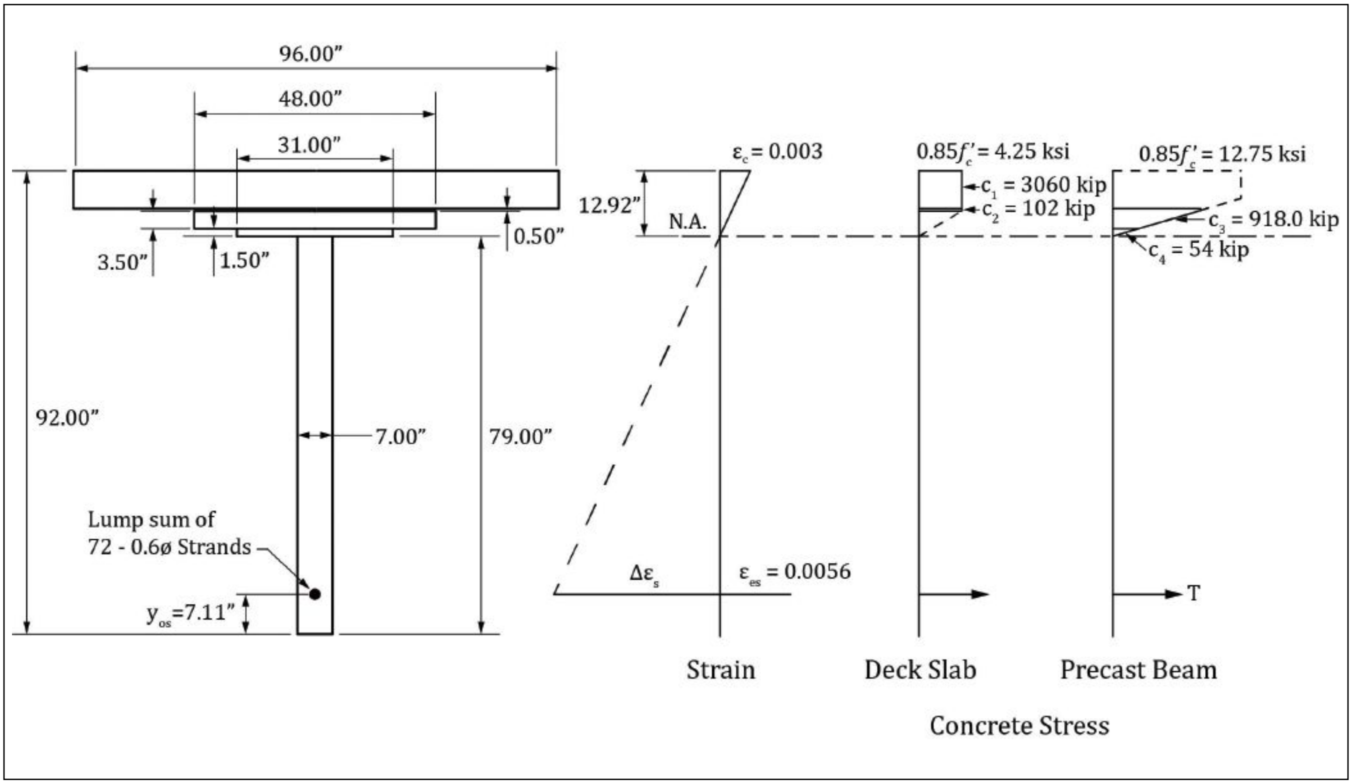

Figure 3. Simplified cross section using rectangular concrete layers and values for final iteration of strain compatibility calculations.

at the service limit state is

f_{pe} = 0.75(270 ksi)(1 – 0.21) = 160 ksi

The strain in the strand at the service limit state is

$$\frac{f_{pe}}{E_{ps}} = \frac{160}{28{,}500} = 0.0056$$

The total strain in the strands (lumped at their centroid) is calculated with the following equation:

$$\varepsilon_s = 0.003\left(\frac{d_{ps}}{c} - 1\right) + \left(\frac{f_{pe}}{E_p}\right) = 0.003\left(\frac{(92 - 7.11)}{12.0} - 1\right) + 0.0056$$

$$= 0.0238$$

The average stress in the prestressing strands is calculated as follows:

$$f_{ps} = 28{,}500(0.0238)\left[0.031 + \frac{0.969}{\left[1 + \left((112.77)(0.0238)\right)^{7.36}\right]^{\frac{1}{7.36}}}\right]$$

$$= 265.9 \text{ ksi} \leq 270.0 \text{ ksi}$$

Tension force $T = f_{ps}A_{ps}$ = (265.9 ksi)(72)(0.217 in.^{2}) = 4154 kips

The compression forces in the deck and haunch buildup are calculated as follows:

Strain at the bilinear transition point for deck concrete is 0.85(5 ksi)/(4287 ksi) = 0.001

When the strain exceeds this value, the stress is taken to be $f_c = 0.85f'_c$

Strain in the deck $$\varepsilon_1 = \frac{0.003\left(12 - \left(\frac{7.5}{2}\right)\right)}{12} = 0.0021 \geq 0.001$$

Stress in the deck f_{c1} = 0.85(5 ksi) = 4.25 ksi

Compression force in the deck C_1 = (4.25 ksi)(96 in.)(7.5 in.) = 3060 kips

Strain in the haunch $$\varepsilon_2 = \frac{0.003\left(12 - \left(7.5 + \frac{0.5}{2}\right)\right)}{12}$$

$$= 0.0011 \geq 0.001$$

Stress in the haunch f_{c2} = 0.85(5 ksi) = 4.25 ksi
Compression force in the haunch C_2 = (4.25 ksi)(48 in.)(0.5 in.) = 102 kips

The compression forces in the two layers of the girder top flange are calculated as follows:

Strain at the bilinear transition point for girder concrete is

$$\frac{0.85(15 \text{ ksi})}{7425 \text{ ksi}} = 0.0017$$

When the strain is less than this value, the stress is taken to be $f_c = E_c \varepsilon_c$

Strain in the upper top flange element

$$\varepsilon_3 = \frac{0.003\left(12-\left(7.5+0.5+\dfrac{3.5}{2}\right)\right)}{12} = 0.0006 < 0.0017$$

Stress in the upper top flange element f_{c3} = 7425 ksi (0.0006) = 4.46 ksi

Compression force in the upper top flange element
C_3 = (4.46 ksi)(48 in.)(3.5 in.) = 749 kips

Strain in the lower top flange element

$$\varepsilon_4 = \frac{0.003\left(12-\left(7.5+0.5+3.5+\dfrac{0.5}{2}\right)\right)}{12} = 0.0001 < 0.0017$$

Stress in the lower top flange element f_{c4} = 7425 ksi (0.0001) = 0.74 ksi

Compression force in the lower top flange element
C_4 = (0.74 ksi)(31 in.)(0.5 in.) = 11 kips

The sum of tension and compression forces is as follows:

$$\Sigma F = T - C_1 - C_2 - C_3 - C_4 = 4154 \text{ kips} - 3060 \text{ kips} - 102 \text{ kips} - 749 \text{ kips} - 11 \text{ kips} = 232 \text{ kips}$$

The tension and compression forces are not in equilibrium. There is excess tension force, which can be remedied by increasing the depth to the neutral axis. The proceeding calculations as well as calculations for the next two iterations are presented in **Table 1**. The next estimate of neutral axis depth is c = 13 in. The sum of the forces using this estimate for c is not yet in equilibrium, but it is close, with an excess compression force. The final iteration is c = 12.92 in., and for this iteration, the tension and compression forces sum to zero, indicating that the correct neutral axis depth has been found.

Finally, to determine nominal strength of the composite beam, the sum of the moments about the top face of the composite section is calculated as follows:

$$M_n = (4134 \text{ kips})(84.89 \text{ in.}) - (3060 \text{ kips})\left(\frac{7.5 \text{ in.}}{2}\right)$$
$$- (102 \text{ kips})\left(7.5 \text{ in.} + \frac{0.5 \text{ in.}}{2}\right)$$
$$- (918 \text{ kips})\left(7.5 \text{ in.} + 0.5 \text{ in.} + \left(\frac{3.5 \text{ in.}}{2}\right)\right)$$
$$- (54 \text{ kips})(7.5 \text{ in.} + 0.5 \text{ in.} + 3.5 \text{ in.} + 0.71 \text{ in.})$$
$$= 329{,}060 \text{ in.-kips} = 27{,}422 \text{ ft-kips}$$

Conclusion

This strain compatibility analysis could have been carried out using a rectangular or nonlinear stress-strain relationship for the concrete, a bilinear relationship for the reinforcement, or different idealizations of the girder and slab elements. Not all strain compatibility analyses use the same assumptions or modeling. Design engineers must understand the methods, assumptions, material properties, and limitations of the strain compatibility and moment-capacity analyses they use, especially those implemented in software tools. The software tools should provide sufficiently detailed information such that the analyses can be understood and validated.

Strain compatibility plays a vital role in various engineering applications. It provides a means to accurately estimate stress and strain in concrete components. Strain compatibility provides engineers with a reliable method of designing concrete structures. Part 2 of this series will discuss the assumptions and limitations of the rectangular stress distribution method specified in the AASHTO LRFD specifications, as well as the use of strain compatibility analysis to overcome those limitations.

References

1. American Association of State Highway and Transportation Officials (AASHTO). 2020. *AASHTO LRFD Bridge Design Specifications.* 9th ed. Washington, DC: AASHTO.
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3. Devalapura, R. K., and M. K. Tadros. 1992. "Stress-Strain Modeling of 270 ksi Low-Relaxation Prestressing Strands." *PCI Journal* 37 (2): 100–106. https://doi.org/10.15554/pcij.03011992.100.106

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Table 1. Strain compatibility analysis for different estimates of neutral axis depth.

Trial	c, in.	ε_s	f_{ps}, ksi	T, kip	ε_1	f_{c1}, ksi	C_1, kip	ε_2	f_{c2}, ksi	C_2, kip	ε_3	f_{c3}, ksi	C_3, kip	ε_4	f_{c4}, ksi	C_4, kip	ΣF, kip
1	12	0.0238	265.9	4154	0.0021	4.25	3060	0.0011	4.25	102	0.0006	4.46	749	0.0001	0.74	11	232
2	13	0.0222	264.5	4132	0.0021	4.25	3060	0.0012	4.25	102	0.0008	5.57	936	0.0002	1.29	60	–26
3	12.92	0.0223	264.6	4134	0.0021	4.25	3060	0.0012	4.25	102	0.0007	5.47	918	0.0002	1.22	54	0